8.05 Quantum Physics II — Course Summary

Zwiebach, *Mastering Quantum Mechanics* Part II (Ch 12–23) + MIT 8.05 notes (2013)

The course in 8 ideas

1. **New language, same physics:** a state is a vector $|\Psi\rangle$ in a complex Hilbert space; the wavefunction is its components, $\psi(x) = \langle x|\Psi\rangle$.
2. **It's all linear algebra:** Hermitian ⇒ real eigenvalues + orthonormal eigenbasis (spectral theorem). Measurement = projection onto an eigenspace.
3. **Compatible observables** commute ⇔ share an eigenbasis; a CSCO labels states uniquely. Non-commuting pairs obey $\Delta A \Delta B \geq |\langle [A,B] \rangle|/2$.
4. **Time evolution is unitary,** $U = e^{-i\hat{H}t/\hbar}$. Schrödinger (states move) and Heisenberg (operators move) pictures give identical predictions.
5. **Spin ½ is the simplest quantum system.** Any two-level system is a spin in an effective field and precesses at $\omega = 2|\vec{g}|/\hbar$.
6. **Oscillator beyond $|n\rangle$:** coherent states move like classical particles with minimal noise; squeezed states trade Δx for Δp.
7. **Composite systems = tensor products.** Entanglement powers teleportation and Bell violations; subsystems are described by density matrices; decoherence is entanglement with the environment.
8. **Angular momentum from commutators alone:** $j = 0, \frac{1}{2}, 1, \ldots$; ladder operators; $j_1 \otimes j_2 = \bigoplus j$ explains hyperfine and spin-orbit structure.

1 · Wave mechanics refresher notes 1 · Ch 6–7

$$i\hbar\,\partial_t \Psi = \Big(-\frac{\hbar^2}{2m}\nabla^2 + V\Big)\Psi, \qquad \Psi = \psi(x)e^{-iEt/\hbar}$$

- 1D bound states: **non-degenerate**, can be chosen **real**; even V ⇒ bound states even/odd; n-th state has $n-1$ nodes; $E > V_{\min}$.
- **Spectrum:** $E < \min(V_{-\infty}, V_{+\infty})$: discrete, non-degenerate. Between the two asymptotes: continuous, non-degenerate. Above both: continuous, doubly degenerate.
- Matching: ψ always continuous; ψ' continuous unless δ-function or infinite wall. $V = -\alpha\delta(x)$: $\Delta\psi'(0) = -\frac{2m\alpha}{\hbar^2}\psi(0)$.

Variational principle. For any normalizable ψ:

$$E_{gs} \leq \frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\int \left(\frac{\hbar^2}{2m}|\psi'|^2 + V|\psi|^2\right)dx}{\int |\psi|^2 dx}$$

Equality iff ψ is the ground state. Error in E is second order in the error in ψ. Use trial functions with parameters, minimize. Any attractive 1D potential has a bound state.

Position & momentum: $\hat{p} = \frac{\hbar}{i}\partial_x$, $\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int e^{-ipx/\hbar}\psi(x)dx$; in p-space $\hat{x} = i\hbar\,\partial_p$.

2 · Spin one-half notes 2 · Ch 12, 14.7

Stern–Gerlach: $\vec{F} = \nabla(\vec{\mu}\cdot\vec{B})$ splits silver atoms into 2 beams ⇒ two-state system. Sequential SG boxes: z-filter then x-filter gives ½/½; states $|z;+\rangle, |z;-\rangle$ orthogonal (not "opposite vectors").

$$\hat{\vec{S}} = \frac{\hbar}{2}\vec{\sigma}, \quad \sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix},\ \sigma_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix},\ \sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$[\hat{S}_i, \hat{S}_j] = i\hbar\epsilon_{ijk}\hat{S}_k, \qquad \hat{S}^2 = \tfrac{3}{4}\hbar^2\,\mathbb{1}$$

$$\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k, \quad [\sigma_i,\sigma_j] = 2i\epsilon_{ijk}\sigma_k, \quad \{\sigma_i,\sigma_j\} = 2\delta_{ij}$$

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b})\,\mathbb{1} + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$$

$$e^{i\theta\,\hat{n}\cdot\vec{\sigma}} = \cos\theta\,\mathbb{1} + i\sin\theta\,\hat{n}\cdot\vec{\sigma} \quad (|\hat{n}| = 1)$$

$\sigma_i^2 = \mathbb{1}$, $\mathrm{tr}\,\sigma_i = 0$, $\det\sigma_i = -1$, eigenvalues ±1, $\mathrm{tr}(\sigma_i\sigma_j) = 2\delta_{ij}$. Any 2×2: $M = a_0\mathbb{1} + \vec{a}\cdot\vec{\sigma}$, $a_0 = \frac{1}{2}\mathrm{tr}M$, $a_i = \frac{1}{2}\mathrm{tr}(M\sigma_i)$.

Spin along $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$: $\hat{S}_n = \hat{n}\cdot\hat{\vec{S}}$

$$|n;+\rangle = \cos\tfrac{\theta}{2}|+\rangle + \sin\tfrac{\theta}{2}\,e^{i\phi}|-\rangle$$

$$|n;-\rangle = \sin\tfrac{\theta}{2}|+\rangle - \cos\tfrac{\theta}{2}\,e^{i\phi}|-\rangle$$

$$|x;\pm\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle), \qquad |y;\pm\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle \pm i|-\rangle)$$

$\langle n;+|\hat{\vec{S}}|n;+\rangle = \frac{\hbar}{2}\hat{n}$; $|\langle n';+|n;+\rangle|^2 = \cos^2\frac{\gamma}{2}$ (γ = angle between $\hat{n}, \hat{n}'$).

Every spin-½ state is $|n;+\rangle$ for some $\hat{n}$ (Bloch sphere). Orthogonal states ↔ antipodal points.

Rotating spin states: $\hat{R}_{\hat{n}}(\alpha) = e^{-i\alpha\hat{n}\cdot\hat{\vec{S}}/\hbar} = \cos\frac{\alpha}{2}\,\mathbb{1} - i\sin\frac{\alpha}{2}\,\hat{n}\cdot\vec{\sigma}$ rotates $\langle\hat{\vec{S}}\rangle$ by α about $\hat{n}$; $\hat{R}(2\pi) = -\mathbb{1}$ (spinors need 4π).

Quantum key distribution (BB84). Alice sends random bits, each encoded in a random basis (z or x). Bob measures in a random basis. They publicly compare *bases* (not bits) and keep the ~½ that match: the sifted key. An eavesdropper who intercepts and resends in a random basis causes **25% errors** in the sifted key, so comparing a sample reveals her. Security rests on no-cloning and disturbance by measurement.

3 · Linear algebra: vector spaces & operators notes 3 · Ch 13–14

- **Basis** = linearly independent spanning list; $\dim V$ = length of any basis. $\mathcal{L}(V)$ = linear maps $V \to V$.
- Matrix of T in basis $\{v_j\}$: $Tv_j = \sum_i T_{ij}v_i$. Change of basis A: $T' = A^{-1}TA$; tr, $\det$ basis-independent.
- $\mathrm{null}\,T$, $\mathrm{range}\,T$; rank–nullity: $\dim V = \dim\mathrm{null}\,T + \dim\mathrm{range}\,T$. Finite dim: injective ⇔ surjective ⇔ invertible ⇔ null $T = \{0\}$.
- **Eigenvalue** λ: $T - \lambda\mathbb{1}$ not invertible ⇔ $\det(T-\lambda\mathbb{1}) = 0$. Every operator on a finite-dim **complex** space has ≥1 eigenvalue. Eigenvectors with distinct eigenvalues are linearly independent.
- **Inner product**: $\langle a,b\rangle$ linear in 2nd slot, $\langle a,b\rangle = \langle b,a\rangle^*$, $\langle a,a\rangle \geq 0$ (=0 iff $a = 0$).

Cauchy–Schwarz: $|\langle u,v\rangle| \leq |u|\,|v|$, **Triangle:** $|u+v| \leq |u| + |v|$

Gram–Schmidt: $e_1 = \frac{v_1}{|v_1|}$, $e_j = \frac{v_j - \sum_{i<j}\langle e_i, v_j\rangle e_i}{|\cdots|}$.

Orthogonal projector onto U: $P_U v = \sum_i \langle e_i, v\rangle e_i$; $P_U^2 = P_U$, $P_U^\dagger = P_U$, $V = U \oplus U^\perp$.

- **Riesz:** every linear functional is $\phi(v) = \langle u, v\rangle$ for a unique u.
- **Adjoint:** $\langle u, Tv\rangle = \langle T^\dagger u, v\rangle$; $(ST)^\dagger = T^\dagger S^\dagger$; $(T^\dagger)_{ij} = T^*_{ji}$ in an orthonormal basis.
- **Hermitian** $T = T^\dagger \Leftrightarrow \langle v, Tv\rangle \in \mathbb{R}\ \forall v$ (complex space). Real eigenvalues; orthogonal eigenvectors.
- Complex space: $\langle v, Tv\rangle = 0\ \forall v \Rightarrow T = 0$. (Fails for real spaces: rotations by 90°.)
- **Unitary** $U^\dagger U = \mathbb{1}$ ⇔ preserves norms ⇔ maps orthonormal bases to orthonormal bases. Eigenvalues $|\lambda| = 1$.
- **Functions of operators:** $f(\hat{A}) = \sum_i f(\lambda_i)P_i$ for diagonalizable $\hat{A}$; power series otherwise. $\frac{d}{dt}e^{t\hat{A}} = \hat{A}e^{t\hat{A}}$; $e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}}$ only if $[\hat{A},\hat{B}] = 0$.

4 · Dirac notation & the axioms notes 4 · Ch 14, 16.6

$$\langle a|b\rangle = \langle a,b\rangle, \quad \langle a|b\rangle^* = \langle b|a\rangle, \quad (\alpha|a\rangle)^\dagger = \alpha^*\langle a|$$

$$\textbf{Completeness:}\ \sum_i |i\rangle\langle i| = \mathbb{1}, \qquad T = \sum_{i,j}|i\rangle\langle i|T|j\rangle\langle j|$$

$$\langle a|T^\dagger|b\rangle = \langle b|T|a\rangle^*, \qquad (|a\rangle\langle b|)^\dagger = |b\rangle\langle a|$$

- Projector onto unit $|u\rangle$: $P = |u\rangle\langle u|$; onto subspace: $P = \sum_{i\in U}|i\rangle\langle i|$, $\mathrm{tr}P = \dim U$.
- Unitary between orthonormal bases: $U = \sum_i |f_i\rangle\langle e_i|$.
- Insert $\mathbb{1}$ anywhere: $\langle a|AB|b\rangle = \sum_i \langle a|A|i\rangle\langle i|B|b\rangle$.

Continuous bases:

$$\hat{x}|x\rangle = x|x\rangle, \quad \langle x|y\rangle = \delta(x-y), \quad \int dx\,|x\rangle\langle x| = \mathbb{1}, \quad \psi(x) = \langle x|\psi\rangle$$

$$\langle x|p\rangle = \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}, \quad \langle x|\hat{p}|\psi\rangle = \frac{\hbar}{i}\frac{d}{dx}\psi(x), \quad \langle p|\hat{x}|\psi\rangle = i\hbar\frac{d}{dp}\tilde{\psi}(p)$$

$$\langle\phi|\psi\rangle = \int \phi^*(x)\psi(x)dx = \int \tilde{\phi}^*(p)\tilde{\psi}(p)dp$$

$|x\rangle$, $|p\rangle$ are not normalizable (not in the Hilbert space) but form complete "bases".

Axioms (any isolated system): **A1** a state is a ray in a Hilbert space ($|\Psi\rangle \sim c|\Psi\rangle$). **A2** observables are Hermitian operators, $\hat{A} = \sum_k a_k P_k$ (spectral decomposition). **A3** measurement along $\{P_k\}$ gives k with $p(k) = \langle\Psi|P_k|\Psi\rangle$; the state becomes $P_k|\Psi\rangle/\sqrt{p(k)}$. **A4** time evolution is unitary. **Postulates:** P1 composite system space = $\mathcal{H}_A \otimes \mathcal{H}_B$. P2 identical particles: only totally symmetric (bosons) or antisymmetric (fermions) states. Open systems → density matrices (§11).

5 · Uncertainty & compatible observables notes 5 · Ch 15

$$\Delta A = \big|(\hat A - \langle\hat A\rangle)\Psi\big|, \qquad (\Delta A)^2 = \langle\hat A^2\rangle - \langle\hat A\rangle^2$$

$$\Delta A\,\Delta B \;\geq\; \left|\left\langle\Psi\middle|\frac{1}{2i}[\hat A,\hat B]\middle|\Psi\right\rangle\right|$$

Saturated iff $(\hat B - \langle\hat B\rangle)\Psi = i\lambda(\hat A - \langle\hat A\rangle)\Psi$, $\lambda\in\mathbb{R}$. For x,p ⇒ Gaussian $\psi \propto e^{-\lambda(x-\langle x\rangle)^2/2\hbar + i\langle p\rangle x/\hbar}$.

Proof sketch: $f = (\hat A - \langle A\rangle)\Psi$, $g = (\hat B - \langle B\rangle)\Psi$; Schwarz $|f||g| \geq |\langle f,g\rangle| \geq |\mathrm{Im}\langle f,g\rangle|$.

Energy–time (Mandelstam–Tamm): for any $\hat Q$ (no explicit t),

$$\Delta H\,\Delta Q \geq \frac{\hbar}{2}\left|\frac{d\langle\hat Q\rangle}{dt}\right| \;\Rightarrow\; \Delta H\,\Delta t \geq \frac{\hbar}{2}, \quad \Delta t \equiv \frac{\Delta Q}{|d\langle Q\rangle/dt|}$$

Decay: lifetime τ ↔ linewidth $\Delta E \sim \hbar/\tau$.

Lower bounds on E_{gs}: write $\langle H\rangle$ in terms of $\langle p^2\rangle \geq (\Delta p)^2$, $\langle x^2\rangle \geq (\Delta x)^2$, $\langle x^4\rangle \geq \langle x^2\rangle^2$, then use $\Delta p \geq \hbar/2\Delta x$ and minimize. E.g. $H = \frac{p^2}{2m} + \alpha x^4 \Rightarrow E_{gs} \geq \frac{3}{8}\left(\frac{2\hbar^4\alpha}{m^2}\right)^{1/3}$. (Upper bound: variational.)

- **Diagonalizable** ⇔ basis of eigenvectors; then $T = ADA^{-1}$. **Unitarily** diagonalizable ⇔ orthonormal eigenbasis.
- **Spectral theorem:** M unitarily diagonalizable ⇔ M is **normal** ($[M, M^\dagger] = 0$). Hermitian & unitary are normal. $M = \sum_k \lambda_k P_k$.
- **Simultaneous diagonalization:** Hermitian A, B have a common orthonormal eigenbasis ⇔ $[A,B] = 0$ (compatible observables). Within each degenerate eigenspace of A, diagonalize B.
- **CSCO:** set of commuting Hermitian operators whose eigenvalues uniquely label every basis state (e.g. $\{H, L^2, L_z\}$ for hydrogen without spin).

6 · Quantum dynamics notes 6 · Ch 16–17

Harmonic oscillator (algebraic)

$$\hat a = \sqrt{\tfrac{m\omega}{2\hbar}}\left(\hat x + \tfrac{i\hat p}{m\omega}\right),\ [\hat a,\hat a^\dagger] = 1,\ \hat H = \hbar\omega(\hat N + \tfrac12),\ \hat N = \hat a^\dagger\hat a$$

$$\hat x = \sqrt{\tfrac{\hbar}{2m\omega}}(\hat a + \hat a^\dagger),\quad \hat p = i\sqrt{\tfrac{m\omega\hbar}{2}}(\hat a^\dagger - \hat a),\quad |n\rangle = \tfrac{(\hat a^\dagger)^n}{\sqrt{n!}}|0\rangle$$

$$\hat a|n\rangle = \sqrt{n}|n-1\rangle,\quad \hat a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle,\quad \langle x|0\rangle \propto e^{-m\omega x^2/2\hbar}$$

Schrödinger picture: unitary time evolution

$$|\Psi,t\rangle = \mathcal{U}(t,t_0)|\Psi,t_0\rangle,\quad \mathcal{U}^\dagger\mathcal{U} = \mathbb{1},\quad \mathcal{U}(t_2,t_1)\mathcal{U}(t_1,t_0) = \mathcal{U}(t_2,t_0)$$

$$i\hbar\,\partial_t\mathcal{U}(t,t_0) = \hat H(t)\mathcal{U}(t,t_0)\quad (\hat H \equiv i\hbar(\partial_t\mathcal{U})\mathcal{U}^\dagger \text{ is Hermitian})$$

(1) $\hat H$ **const:** $\mathcal{U} = e^{-i\hat H(t-t_0)/\hbar}$

(2) $[\hat H(t_1),\hat H(t_2)] = 0$**:** $\mathcal{U} = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\hat H(t')dt'\right]$

(3) general: $\mathcal{U} = \mathcal{T}\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\hat H(t')dt'\right]$ (time-ordered, Dyson serie

Heisenberg picture

$$\hat A_H(t) = \mathcal{U}^\dagger(t,0)\,\hat A_S\,\mathcal{U}(t,0),\qquad \langle\Psi,t|\hat A_S|\Psi,t\rangle = \langle\Psi,0|\hat A_H(t)|\Psi,0\rangle$$

$$i\hbar\frac{d\hat A_H}{dt} = [\hat A_H,\hat H_H] + i\hbar\Big(\frac{\partial\hat A_S}{\partial t}\Big)_H,\qquad [\hat A,\hat B]_H = [\hat A_H,\hat B_H]$$

$\hat H_H = \hat H_S$ if $\hat H_S$ is time-independent. Products: $(\hat A\hat B)_H = \hat A_H\hat B_H$.

System	Heisenberg operators
Free particle	$\hat x_H = \hat x + \frac{\hat p}{m}t$, $\hat p_H = \hat p$
Oscillator	$\hat x_H = \hat x\cos\omega t + \frac{\hat p}{m\omega}\sin\omega t$ $\hat p_H = \hat p\cos\omega t - m\omega\hat x\sin\omega t$ $\hat a_H = e^{-i\omega t}\hat a$, $\hat a_H^\dagger = e^{i\omega t}\hat a^\dagger$
Linear $V = -Fx$	$\hat p_H = \hat p + Ft$, $\hat x_H = \hat x + \frac{\hat p}{m}t + \frac{F}{2m}t^2$

Coherent states

Translation: $\hat T_{x_0} = e^{-i\hat p x_0/\hbar}$, $\hat T_{x_0}^\dagger\hat x\,\hat T_{x_0} = \hat x + x_0$, $\langle x|\hat T_{x_0}|\psi\rangle = \psi(x -$

$$|\alpha\rangle = \hat D(\alpha)|0\rangle = e^{\alpha\hat a^\dagger - \alpha^*\hat a}|0\rangle = e^{-|\alpha|^2/2}\sum_n\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$\hat a|\alpha\rangle = \alpha|\alpha\rangle,\qquad \alpha = \frac{\langle\hat x\rangle}{\sqrt2\,d} + i\frac{\langle\hat p\rangle\,d}{\sqrt2\,\hbar},\quad d = \sqrt{\tfrac{\hbar}{m\omega}}$$

$$|\alpha,t\rangle = e^{-i\omega t/2}\,|\alpha e^{-i\omega t}\rangle\quad \text{(stays coherent)}$$

- $\langle x\rangle(t)$, $\langle p\rangle(t)$ follow the classical orbit; $\Delta x = \frac{d}{\sqrt2}$, $\Delta p = \frac{\hbar}{\sqrt2 d}$ (vacuum values, minimum uncertainty) for all t.
- Number distribution is **Poisson**: $P(n) = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}$, $\langle N\rangle = |\alpha|^2$, $\Delta N = |\alpha|$.
- Not orthogonal: $|\langle\beta|\alpha\rangle|^2 = e^{-|\alpha-\beta|^2}$; overcomplete. $\hat D^\dagger(\alpha)\hat a\hat D(\alpha) = \hat a + \alpha$.
- Real α: $\hat D(\alpha) = \hat T_{x_0}$ with $x_0 = \sqrt2\,d\,\alpha$.

BCH: if $[\hat A,\hat B]$ commutes with $\hat A$ and $\hat B$: $e^{\hat A+\hat B} = e^{\hat A}e^{\hat B}e^{-\frac12[\hat A,\hat B]}$.

$$e^{\hat A}\hat B e^{-\hat A} = \hat B + [\hat A,\hat B] + \tfrac{1}{2!}[\hat A,[\hat A,\hat B]] + \cdots$$

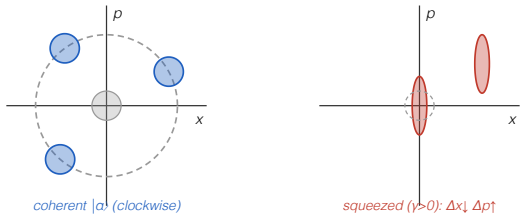
Squeezed states

$$\hat S(\gamma) = \exp\Big[-\frac{\gamma}{2}\big(\hat a^\dagger\hat a^\dagger - \hat a\hat a\big)\Big],\quad |0_\gamma\rangle = \hat S(\gamma)|0\rangle = \frac{1}{\sqrt{\cosh\gamma}}e^{-\frac12\tanh\gamma\,\hat a}$$

$$\Delta x = e^{-\gamma}\sqrt{\tfrac{\hbar}{2m\omega}},\qquad \Delta p = e^{\gamma}\sqrt{\tfrac{m\omega\hbar}{2}},\qquad \Delta x\Delta p = \tfrac{\hbar}{2}$$

$(\hat a\cosh\gamma + \hat a^\dagger\sinh\gamma)|0_\gamma\rangle = 0$. Contains only even n. General: $\hat D(\alpha)\hat S(\gamma)|0\rangle$.

Origin: ground state of an oscillator with frequency ω_1, written in terms of a different oscillator ω_2; $e^\gamma = \sqrt{\omega_1/\omega_2}$. Uncertainty ellipse rotates at 2ω under time evolution.



Phase space ($\langle x\rangle/d$, $\langle p\rangle d/\hbar$): coherent state = vacuum disc carried around a classical circle; squeezed state = ellipse with the same area.

Photon states: each EM mode ↔ oscillator; $\hat{\vec E} \propto (\hat a e^{-i\omega t} + \hat a^\dagger e^{i\omega t})$; $|n\rangle$ = n photons, $\langle\hat{\vec E}\rangle = 0$; coherent state = classical wave; squeezed light reduces noise in one quadrature (LIGO).

Factorization method (Ch 17.8). Write $\hat H = \hat A^\dagger\hat A + E_0$ with $\hat A = \frac{\hbar}{\sqrt{2m}}\left(\frac{d}{dx} + W(x)\right)$, so $V = \frac{\hbar^2}{2m}(W^2 - W') + E_0$. Then $\hat A\psi_0 = 0$ gives the ground state $\psi_0 \propto e^{-\int W}$, energy E_0. The partner $\hat H' = \hat A\hat A^\dagger + E_0$ has the **same spectrum minus E_0**, and $\hat A$ maps eigenstates of $\hat H$ to those of $\hat H'$. SHO: $W = \frac{m\omega}{\hbar}x$. Works for the hydrogen radial equation too.

7 · Two-state systems notes 7 · Ch 17

$$\vec\mu = \gamma\hat{\vec S},\quad \gamma = \frac{gq}{2m},\quad \hat H = -\vec\mu\cdot\vec B = -\gamma\vec B\cdot\hat{\vec S}$$

Electron: $g \approx 2$, $\vec\mu = -\frac{e}{m}\hat{\vec S}$, $\mu_B = \frac{e\hbar}{2m}$.

Larmor: $\hat H = \vec\omega_L\cdot\hat{\vec S},\quad \vec\omega_L = -\gamma\vec B,\qquad \frac{d\langle\hat{\vec S}\rangle}{dt} = \vec\omega_L\times\langle\hat{\vec S}\rangle$

$\hat R_{\hat n}(\alpha) = e^{-i\alpha\hat n\cdot\hat{\vec S}/\hbar}$ **rotates spin states by α about $\hat n$**

$\mathcal{U}(t) = \hat R_{\hat\omega_L}(\omega_L t)$. Note $\hat R_{\hat n}(2\pi) = -\mathbb{1}$ for spin ½.

Any two-state Hamiltonian is a spin in a field:

$$\hat H = g_0\mathbb{1} + \vec g\cdot\vec\sigma = g_0\mathbb{1} + \frac{2}{\hbar}\vec g\cdot\hat{\vec S} \;\Rightarrow\; \vec\omega = \frac{2\vec g}{\hbar}$$

Energies $g_0 \pm |\vec g|$, eigenstates $|n;\pm\rangle$ with $\hat n = \vec g/|\vec g|$. Map state $\alpha|1\rangle + \beta|2\rangle$ to a Bloch vector and precess it about $\vec\omega$.

Ammonia molecule

Basis $|\uparrow\rangle, |\downarrow\rangle$ (N above/below H plane); tunneling couples them:

$$\hat H = \begin{pmatrix} E_0 & -\Delta \\ -\Delta & E_0\end{pmatrix}$$

$$|G\rangle = \tfrac{1}{\sqrt2}(|\uparrow\rangle + |\downarrow\rangle),\ E_0 - \Delta;\qquad |E\rangle = \tfrac{1}{\sqrt2}(|\uparrow\rangle - |\downarrow\rangle),\ E_0 + \Delta$$

$2\Delta \approx 0.987\times10^{-4}$ eV, $\nu = 23.87$ GHz ($\lambda \approx 1.26$ cm). Starting in $|\uparrow\rangle$: oscillates to $|\downarrow\rangle$ at frequency $2\Delta/\hbar$.

In a static field $\mathcal{E}$ (dipole μ): $\hat H = \begin{pmatrix} E_0 + \mu\mathcal{E} & -\Delta \\ -\Delta & E_0 - \mu\mathcal{E}\end{pmatrix}$,

$E_\pm = E_0 \pm \sqrt{\Delta^2 + \mu^2\mathcal{E}^2}$. Field gradient separates $|E\rangle$, $|G\rangle$ beams.

Maser: $\mathcal{E}(t) = 2\mathcal{E}_0\cos\omega_0 t$, $\hbar\omega_0 = 2\Delta$; drop $e^{\pm2i\omega_0 t}$ terms (rotating-wave approx.) ⇒ $P_{E\to G}(t) = \sin^2\left(\frac{\mu\mathcal{E}_0 t}{\hbar}\right)$.

Nuclear magnetic resonance

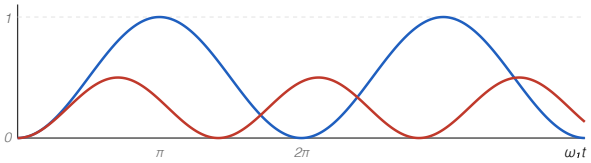
$$\vec B(t) = B_0\hat z + B_1(\hat x\cos\omega t - \hat y\sin\omega t),\qquad \omega_0 = \gamma B_0,\ \omega_1 = \gamma B_1$$

Rotating frame $|\Psi_R\rangle = e^{-i\omega t\hat S_z/\hbar}|\Psi\rangle$ gives a **time-independent** field

$$\vec B_R = B_1\hat x + B_0\Big(1 - \frac{\omega}{\omega_0}\Big)\hat z,\qquad |\Psi,t\rangle = e^{i\omega t\hat S_z/\hbar}e^{i\gamma\vec B_R\cdot\hat{\vec S}t/\hbar}|\Psi,0\rangle$$

Resonance $\omega = \omega_0$: $\vec B_R = B_1\hat x$, spin rotates slowly (ω_1) from z toward the plane while precessing fast (ω_0). Spin-flip prob. (Rabi): $P = \frac{\omega_1^2}{\Omega^2}\sin^2\frac{\Omega t}{2}$, $\Omega = \sqrt{(\omega-\omega_0)^2 + \omega_1^2}$.

90° pulse ($\omega_1 t = \pi/2$) tips spins into xy-plane; precessing magnetization induces signal. Relaxation T_1 (longitudinal), T_2 (transverse) give MRI contrast. Proton: $\gamma/2\pi \approx 42.6$ MHz/T.



Rabi oscillations of spin-flip probability: on resonance (blue) full flips; detuned $|\omega-\omega_0|=\omega_1$ (red) max ½, faster.

8 · Tensor products & entanglement notes 8 · Ch 18

- $V\otimes W$: spanned by $v\otimes w$; $\dim=\dim V\cdot\dim W$. Bilinear: $(av_1+bv_2)\otimes w=a\,v_1\otimes w+b\,v_2\otimes w$.
- $(\hat S\otimes\hat T)(v\otimes w)=\hat S v\otimes\hat T w$; $\hat S\otimes\mathbb{1}$ and $\mathbb{1}\otimes\hat T$ commute. $\langle v_1\otimes w_1|v_2\otimes w_2\rangle=\langle v_1|v_2\rangle\langle w_1|w_2\rangle$.
- **Entangled** = not a product $v\otimes w$. Two qubits $a|{+}{+}\rangle+b|{+}{-}\rangle+c|{-}{+}\rangle+d|{-}{-}\rangle$ is a product state $\Leftrightarrow ad-bc=0$.
- Measuring particle 1 of $\sum_i|i\rangle_1\otimes|\phi_i\rangle_2$ with result i leaves 2 in $|\phi_i\rangle$ (normalized); prob $=\langle\phi_i|\phi_i\rangle$.

Bell basis $|\Phi_i\rangle=(\mathbb{1}\otimes\sigma_i)|\Phi_0\rangle$, orthonormal, all maximally entangled:

$$|\Phi_0\rangle=\tfrac{1}{\sqrt2}(|{+}{+}\rangle+|{-}{-}\rangle),\quad |\Phi_1\rangle=\tfrac{1}{\sqrt2}(|{+}{-}\rangle+|{-}{+}\rangle)$$
$$|\Phi_2\rangle=\tfrac{i}{\sqrt2}(|{+}{-}\rangle-|{-}{+}\rangle),\quad |\Phi_3\rangle=\tfrac{1}{\sqrt2}(|{+}{+}\rangle-|{-}{-}\rangle)$$
$$|{+}{+}\rangle=\tfrac{1}{\sqrt2}(|\Phi_0\rangle+|\Phi_3\rangle),\quad |{-}{-}\rangle=\tfrac{1}{\sqrt2}(|\Phi_0\rangle-|\Phi_3\rangle)$$
$$|{+}{-}\rangle=\tfrac{1}{\sqrt2}(|\Phi_1\rangle-i|\Phi_2\rangle),\quad |{-}{+}\rangle=\tfrac{1}{\sqrt2}(|\Phi_1\rangle+i|\Phi_2\rangle)$$

Teleportation. Alice holds unknown $|\Psi\rangle_C=\alpha|+\rangle+\beta|-\rangle$ and A of a shared $|\Phi_0\rangle_{AB}$. Rewriting,

$$|\Psi\rangle_C\otimes|\Phi_0\rangle_{AB}=\tfrac12\sum_{i=0}^{3}|\Phi_i\rangle_{CA}\otimes\sigma_i|\Psi\rangle_B\quad\textbf{(up to phases)}$$

Alice does a Bell measurement on CA (result i, 2 classical bits) → Bob applies σ_i → B is in $|\Psi\rangle$. Original destroyed (no-cloning); no FTL signalling.

EPR & Bell. Singlet $|\psi\rangle=\frac{1}{\sqrt2}(|{+}{-}\rangle-|{-}{+}\rangle)$ (rotation invariant):

$$\langle(\vec a\cdot\vec\sigma)\otimes(\vec b\cdot\vec\sigma)\rangle=-\vec a\cdot\vec b,\qquad P(\vec a{+},\vec b{+})=\tfrac12\sin^2\tfrac{\theta_{ab}}{2}$$

Local hidden variables ⇒ **Bell inequality** $P(a,b)\le P(a,c)+P(c,b)$. QM with coplanar a,c,b at θ,θ: $\sin^2\theta\le 2\sin^2\frac{\theta}{2}$ — **violated** for small θ. Experiments side with QM.

- **Traces:** $\mathrm{tr}(\hat A\otimes\hat B)=\mathrm{tr}\hat A\,\mathrm{tr}\hat B$. Partial trace $\mathrm{tr}_B(\hat A\otimes\hat B)=\hat A\,\mathrm{tr}\hat B$, extended linearly.
- **No-cloning:** no unitary $\mathcal U|\psi\rangle|e\rangle=|\psi\rangle|\psi\rangle$ for all $|\psi\rangle$. Unitarity preserves $\langle\phi|\psi\rangle$, but cloning would need $\langle\phi|\psi\rangle=\langle\phi|\psi\rangle^2$, true only for identical or orthogonal states.
- Identical particles (Ch 21) are summarized on the 8.06 sheet.

9 · Angular momentum notes 9 · Ch 19

$$\hat{\vec L}=\hat{\vec r}\times\hat{\vec p}=-\hat{\vec p}\times\hat{\vec r},\quad \hat{\vec L}\times\hat{\vec L}=i\hbar\hat{\vec L},\quad \hat{\vec r}\cdot\hat{\vec L}=\hat{\vec p}\cdot\hat{\vec L}=0$$
$$[\hat L_i,\hat V_j]=i\hbar\epsilon_{ijk}\hat V_k\ \ (\hat{\vec V}\text{ vector: }\hat{\vec r},\hat{\vec p},\hat{\vec L}),\qquad [\hat L_i,\hat S]=0\ \ (\hat S\text{ scalar})$$
$$\hat{\vec L}^2=\hat r^2\hat p^2-(\hat{\vec r}\cdot\hat{\vec p})^2+i\hbar\hat{\vec r}\cdot\hat{\vec p}\ \Rightarrow\ \hat p^2=-\hbar^2\frac{1}{r}\partial_r^2 r+\frac{\hat{\vec L}^2}{r^2}$$

Vector identity trick: $(\vec a\times\vec b)_i=\epsilon_{ijk}a_jb_k$, $\epsilon_{ijk}\epsilon_{ipq}=\delta_{jp}\delta_{kq}-\delta_{jq}\delta_{kp}$; keep operator order!

Algebraic theory ($\hat J$ any angular momentum: $[\hat J_i,\hat J_j]=i\hbar\epsilon_{ijk}\hat J_k$)

$$\hat J_\pm=\hat J_x\pm i\hat J_y,\quad [\hat J_z,\hat J_\pm]=\pm\hbar\hat J_\pm,\quad [\hat J_+,\hat J_-]=2\hbar\hat J_z,\quad [\hat J^2,\hat J_i]=0$$
$$\hat J^2=\hat J_+\hat J_-+\hat J_z^2-\hbar\hat J_z=\hat J_-\hat J_++\hat J_z^2+\hbar\hat J_z$$
$$\hat J^2|j,m\rangle=\hbar^2j(j+1)|j,m\rangle,\qquad \hat J_z|j,m\rangle=\hbar m|j,m\rangle$$
$$\hat J_\pm|j,m\rangle=\hbar\sqrt{j(j+1)-m(m\pm1)}\,|j,m\pm1\rangle=\hbar\sqrt{(j\mp m)(j\pm m+1}$$

$j\in\{0,\frac12,1,\frac32,\dots\}$, $m=-j,\dots,j$ ($2j+1$ states). Orbital: only integer ℓ (single-valued $e^{im\phi}$).

Spherical harmonics: $Y_{\ell m}(\theta,\phi)=\langle\hat n|\ell m\rangle$; $L_\pm=\hbar e^{\pm i\phi}(\pm\partial_\theta+i\cot\theta\,\partial_\phi)$; $L_+Y_{\ell\ell}=0\Rightarrow Y_{\ell\ell}\propto\sin^\ell\theta\,e^{i\ell\phi}$. Parity $(-1)^\ell$; $Y_{\ell,-m}=(-1)^mY^*_{\ell m}$; $Y_{\ell0}=\sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta)$. Cartesian: degree-$\ell$ homogeneous harmonic polynomials.

Radial equation $\psi=\frac{u_{E\ell}(r)}{r}Y_{\ell m}$:

$$-\frac{\hbar^2}{2m}u''+\Big(V+\frac{\hbar^2\ell(\ell+1)}{2mr^2}\Big)u=Eu,\quad u(0)=0,\quad u\sim r^{\ell+1}\ (r\to0)$$

Free particle & spherical well

$\rho=kr$: $u=\rho\,j_\ell(\rho)$ (regular) or $\rho\,n_\ell(\rho)$.

$$j_0=\frac{\sin\rho}{\rho},\quad j_1=\frac{\sin\rho}{\rho^2}-\frac{\cos\rho}{\rho},\quad n_0=-\frac{\cos\rho}{\rho}$$
$$j_\ell\xrightarrow{\rho\to0}\frac{\rho^\ell}{(2\ell+1)!!},\qquad j_\ell\xrightarrow{\rho\to\infty}\frac{\sin(\rho-\ell\pi/2)}{\rho}$$
$$e^{ikz}=\sum_\ell(2\ell+1)\,i^\ell j_\ell(kr)P_\ell(\cos\theta)$$

Infinite spherical well (radius a): $E_{n\ell}=\frac{\hbar^2z_{n\ell}^2}{2ma^2}$, $z_{n\ell}$ = n-th zero of j_ℓ.

$z_{n\ell}$	$\ell=0$	1	2	3
$n=1$	3.142	4.493	5.763	6.988
$n=2$	6.283	7.725	9.095	10.417

3D isotropic oscillator: $\hat H=\hbar\omega(\hat N_1+\hat N_2+\hat N_3+\frac32)$, $E=\hbar\omega(N+\frac32)$, degeneracy $\frac12(N+1)(N+2)$, $\ell=N,N-2,\dots,1\text{ or }0$. E.g. $N=2$: $\ell=2\oplus0$ (6 states).

Hydrogen & Runge–Lenz ($V=-e^2/r$):

$$\hat{\vec R}=\frac{1}{2me^2}(\hat{\vec p}\times\hat{\vec L}-\hat{\vec L}\times\hat{\vec p})-\frac{\hat{\vec r}}{r},\qquad[\hat{\vec R},\hat H]=0$$
$$\hat{\vec L}\cdot\hat{\vec R}=\hat{\vec R}\cdot\hat{\vec L}=0,\qquad \hat{\vec R}^2=1+\frac{2\hat H}{me^4}(\hat{\vec L}^2+\hbar^2)$$

$E_n=-\frac{me^4}{2\hbar^2n^2}=-\frac{13.6\,\mathrm{eV}}{n^2}$, $n=N+\ell+1$, $\ell\le n-1$, degeneracy n^2 (×2 with spin). Classically $\vec R$ points to perihelion, $|\vec R|$ = eccentricity.

10 · Addition of angular momentum notes 10 · Ch 20

$$\hat{\vec J}=\hat{\vec J}^{(1)}\otimes\mathbb{1}+\mathbb{1}\otimes\hat{\vec J}^{(2)},\qquad[\hat J_i^{(1)},\hat J_j^{(2)}]=0$$
$$j_1\otimes j_2=(j_1+j_2)\oplus(j_1+j_2-1)\oplus\cdots\oplus|j_1-j_2|$$

Check: $(2j_1+1)(2j_2+1)=\sum_j(2j+1)$. Uncoupled basis $\{J_1^2,J_{1z},J_2^2,J_{2z}\}$ ↔ coupled basis $\{J_1^2,J_2^2,J^2,J_z\}$; $m=m_1+m_2$.

Recipe: top state $|j_1{+}j_2,j_1{+}j_2\rangle=|j_1j_1\rangle|j_2j_2\rangle$; apply $\hat J_-=\hat J_-^{(1)}+\hat J_-^{(2)}$ to get the multiplet; the state orthogonal to it at the next m is the top of the next multiplet; repeat.

½ ⊗ ½ = 1 ⊕ 0

$$|1,1\rangle=|{\uparrow\uparrow}\rangle,\quad |1,0\rangle=\tfrac{1}{\sqrt2}(|{\uparrow\downarrow}\rangle+|{\downarrow\uparrow}\rangle),\quad |1,-1\rangle=|{\downarrow\downarrow}\rangle$$
$$|0,0\rangle=\tfrac{1}{\sqrt2}(|{\uparrow\downarrow}\rangle-|{\downarrow\uparrow}\rangle)$$
$$\hat{\vec S}_1\cdot\hat{\vec S}_2=\tfrac12(\hat S^2-\hat S_1^2-\hat S_2^2)=\begin{cases}+\frac14\hbar^2 & \textbf{triplet}\\ -\frac34\hbar^2 & \textbf{singlet}\end{cases}$$

$\hat{\vec S}_1\cdot\hat{\vec S}_2=\frac12(\hat S_{1+}\hat S_{2-}+\hat S_{1-}\hat S_{2+})+\hat S_{1z}\hat S_{2z}$. Triplet symmetric, singlet antisymmetric under 1↔2.

1 ⊗ ½ = 3/2 ⊕ 1/2

$$|\tfrac32,\tfrac32\rangle=|1,1\rangle|\uparrow\rangle,\quad |\tfrac32,\tfrac12\rangle=\sqrt{\tfrac23}|1,0\rangle|\uparrow\rangle+\sqrt{\tfrac13}|1,1\rangle|\downarrow\rangle$$
$$|\tfrac12,\tfrac12\rangle=\sqrt{\tfrac13}|1,0\rangle|\uparrow\rangle-\sqrt{\tfrac23}|1,1\rangle|\downarrow\rangle\quad\textbf{(overall sign is convention)}$$

General $\ell\otimes\frac12$:

$$|j{=}\ell{\pm}\tfrac12,m\rangle=\sqrt{\tfrac{\ell\pm m+\frac12}{2\ell+1}}|m-\tfrac12\rangle|\uparrow\rangle\pm\sqrt{\tfrac{\ell\mp m+\frac12}{2\ell+1}}|m+\tfrac12\rangle|\downarrow\rangle.$$

Feynman–Hellmann: $\hat H(\lambda)\psi(\lambda)=E(\lambda)\psi(\lambda)\Rightarrow$

$$\frac{dE}{d\lambda}=\Big\langle\psi(\lambda)\Big|\frac{d\hat H}{d\lambda}\Big|\psi(\lambda)\Big\rangle.$$

⇒ first-order shift $\Delta E=\langle\psi^{(0)}|\Delta\hat H|\psi^{(0)}\rangle$ (non-degenerate, or in a basis that diagonalizes $\Delta\hat H$ within the degenerate space).

Hyperfine splitting (H ground state): $\Delta\hat H\propto\hat{\vec S}_e\cdot\hat{\vec S}_p\,|\psi_{100}(0)|^2$

$$E_{\text{triplet}}-E_{\text{singlet}}=\tfrac43\,g_p\frac{m_e}{m_p}\alpha^4m_ec^2\approx5.88\times10^{-6}\,\mathrm{eV}$$

$\Rightarrow\nu\approx1420$ MHz, $\lambda\approx21$ cm (radio astronomy). Singlet ($F=0$) is lower; $g_p\approx5.59$.

Spin–orbit coupling (with Thomas factor ½):

$$\Delta\hat H=\frac{e^2}{2m^2c^2}\frac{1}{r^3}\hat{\vec S}\cdot\hat{\vec L},\qquad\hat{\vec L}\cdot\hat{\vec S}=\tfrac12\big(\hat J^2-\hat L^2-\hat S^2\big)$$

$[\Delta\hat H,\hat L_z]\neq0\Rightarrow$ use CSCO $\{\hat H_0,\hat L^2,\hat S^2,\hat J^2,\hat J_z\}$. $j=\ell\pm\frac12$:

$$\langle\hat{\vec L}\cdot\hat{\vec S}\rangle=\frac{\hbar^2}{2}\big[j(j+1)-\ell(\ell+1)-\tfrac34\big]=\begin{cases}\frac{\hbar^2\ell}{2} & j=\ell+\frac12\\ -\frac{\hbar^2(\ell+1)}{2} & j=\ell-\frac12\end{cases}$$

Splitting ~ α^4mc^2 (fine structure). Notation $n\ell_j$: $2p_{3/2}$, $2p_{1/2}$. Trace check: $\sum_j(2j+1)\langle\vec L\cdot\vec S\rangle=0$.

Hydrogen hidden symmetry: on states of energy E_n, define $\hat{\vec J}_{1,2}=\frac12(\hat{\vec L}\pm\hat{\vec R})$ with $\hat{\vec R}=\sqrt{-\frac{me^4}{2E}}\,\hat{\vec R}$. These are two commuting angular momenta with $j_1=j_2=j$, $n=2j+1$:

$$j\otimes j=0\oplus1\oplus\cdots\oplus2j$$
$$\Rightarrow\ \ell=0,\dots,n-1,\quad\textbf{degeneracy }(2j+1)^2=n^2$$

11 · Density matrix & decoherence Ch 22

Why: a state vector can't describe a subsystem of an entangled pair, or a classical mixture from an imperfect source. The density matrix handles both.

$$\hat\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \qquad \hat\rho^\dagger = \hat\rho, \quad \hat\rho \geq 0, \quad \operatorname{tr}\hat\rho = 1$$

$$\langle \hat A \rangle = \operatorname{tr}(\hat\rho \hat A), \qquad p(k) = \operatorname{tr}(P_k \hat\rho), \qquad i\hbar \frac{d\hat\rho}{dt} = [\hat H, \hat\rho]$$

Pure $\Leftrightarrow \hat\rho^2 = \hat\rho \Leftrightarrow \operatorname{tr}\hat\rho^2 = 1$; mixed: $\operatorname{tr}\hat\rho^2 < 1$. Different ensembles can give the same $\hat\rho$, and then no measurement can tell them apart. **Qubit:** $\hat\rho = \frac{1}{2}(\mathbb{1} + \vec a \cdot \vec\sigma)$, $|\vec a| \leq 1$: surface of the Bloch ball = pure, interior = mixed, centre = maximally mixed.

Subsystems: $\hat\rho_A = \operatorname{tr}_B \hat\rho_{AB}$ gives every prediction for measurements on A alone.
Schmidt decomposition: any $|\psi\rangle_{AB} = \sum_i \sqrt{p_i}\,|a_i\rangle|b_i\rangle$ (orthonormal sets). $\hat\rho_A$ and $\hat\rho_B$ share eigenvalues p_i. Entangled $\Leftrightarrow$ Schmidt rank > 1. Entanglement entropy $S = -\sum_i p_i \ln p_i$ (singlet: $\ln 2$, maximal).

Open systems & decoherence: system S becomes entangled with environment E, so $\hat\rho_S$ loses its off-diagonal terms in a preferred (pointer) basis: superposition → effectively classical mixture. Markovian evolution is the **Lindblad equation**

$$\frac{d\hat\rho}{dt} = -\frac{i}{\hbar}[\hat H, \hat\rho] + \sum_k \left(\hat L_k \hat\rho \hat L_k^\dagger - \tfrac{1}{2}\{\hat L_k^\dagger \hat L_k, \hat\rho\} \right)$$

e.g. dephasing $\hat L = \sqrt{\gamma}\,\sigma_z$ shrinks the off-diagonal ρ_{01} by $e^{-2\gamma t}$; decay $\hat L = \sqrt{\Gamma}\,\sigma_-$ relaxes to $|0\rangle$.

Measurement problem: decoherence explains why we never see superpositions of pointer states, but not why one particular outcome occurs. Interpretations (Copenhagen, many-worlds, …) differ on that point.

12 · Quantum computation Ch 23

Qubits & gates: $|0\rangle, |1\rangle$; n qubits span 2^n dimensions. Gates are unitaries:

$$X = \sigma_1, \quad Z = \sigma_3, \quad H = \frac{1}{\sqrt2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \text{CNOT}|x\rangle|y\rangle = |x\rangle|y \oplus x\rangle$$

$H|0\rangle = |+\rangle$, $H|1\rangle = |-\rangle$; $H^2 = \mathbb{1}$; $HXH = Z$. H then CNOT on $|00\rangle$ makes the Bell state $|\Phi_0\rangle$.

Oracle: $U_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$. With the target in $|-\rangle$: $U_f|x\rangle|-\rangle = (-1)^{f(x)}|x\rangle|-\rangle$ (**phase kickback**).
Deutsch: $f : \{0,1\} \to \{0,1\}$. Prepare $|+\rangle|-\rangle$, apply U_f, then H on qubit 1: it reads 0 if f is constant and 1 if balanced. One query instead of two: interference extracts a global property.

Grover search (N items, one marked $|w\rangle$, uniform start $|s\rangle$): repeat oracle $\mathbb{1} - 2|w\rangle\langle w|$ then diffusion $2|s\rangle\langle s| - \mathbb{1}$. Each round rotates the state by 2θ toward $|w\rangle$ in the $(|w\rangle, |s\rangle)$ plane, $\sin\theta = 1/\sqrt{N}$:

$$P_{\text{success}}(k) = \sin^2\big((2k+1)\theta\big),$$

$$k_{\text{opt}} \simeq \frac{\pi}{4}\sqrt{N} \quad (\text{vs } \sim N/2 \text{ classically})$$

Overshooting lowers the success probability.

13 · Toolkit, constants & pitfalls

$$\int e^{-ax^2+bx}dx = \sqrt{\tfrac{\pi}{a}}e^{b^2/4a}, \qquad \delta(x) = \tfrac{1}{2\pi}\int e^{ikx}dk$$

$$[\hat A, \hat B\hat C] = [\hat A, \hat B]\hat C + \hat B[\hat A, \hat C]$$

$$[\hat x, f(\hat p)] = i\hbar f'(\hat p), \quad [\hat p, f(\hat x)] = -i\hbar f'(\hat x)$$

$$[\hat a, f(\hat a^\dagger)] = f'(\hat a^\dagger), \qquad e^{\lambda \hat N}\hat a\, e^{-\lambda\hat N} = e^{-\lambda}\hat a$$

$$\operatorname{tr}(AB) = \operatorname{tr}(BA), \quad \det e^A = e^{\operatorname{tr}A}, \quad e^{i\theta\sigma_3} = \operatorname{diag}(e^{i\theta}, e^{-i\theta})$$

$\hbar c$	197.3 MeV·fm = 1973 eV·Å	α	1/137.04
$m_e c^2$	0.511 MeV	m_p/m_e	1836
μ_B	5.79×10^{-5} eV/T	a_0	0.529 Å
$\hbar$	6.58×10^{-16} eV·s	Ry	13.6 eV

Strategy: (1) find the CSCO & symmetries first; (2) work in the basis where the perturbation is diagonal (coupled vs uncoupled); (3) two-level problem? → write $g_0\mathbb{1} + \vec g \cdot \vec\sigma$ and think precession; (4) oscillator problem? → go to $\hat a, \hat a^\dagger$; (5) time-dependent H? → try a rotating frame; (6) entanglement question? → test $ad - bc$ / expand in Bell basis.

Common mistakes: $\langle a|T|b\rangle^* = \langle b|T^\dagger|a\rangle$ (dagger!); forgetting $e^{-i\omega t/2}$ zero-point phase; $\hat R(2\pi) = -1$ for spinors; mixing $\vec\omega_L = -\gamma\vec B$ sign; $|n;\pm\rangle$ uses $\theta/2$; $\hat J_\pm$ coefficient $\sqrt{(j\mp m)(j\pm m+1)}$; Heisenberg EOM sign $[\hat A, \hat H]$; $\hat{\vec L}\cdot\hat{\vec S} \neq \hat L_z \hat S_z$; tensor-product ordering of factors.